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Entropy-Driven Phase Transition in a Polydisperse Hard-Rods Lattice System

D. Ioffe,[1] Y. Velenik[2] and M. Zahradník[3]

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We study a system of rods on $\mathbb{Z}^2$, with hard-core exclusion. Each rod has a length between 2 and N. We show that, when N is sufficiently large, and for suitable fugacity, there are several distinct Gibbs states, with orientational long-range order. This is in sharp contrast with the case $N = 2$ (the monomer-dimer model), for which Heilmann and Lieb proved absence of phase transition at any fugacity. This is the first example of a pure hard-core system with phases displaying orientational order, but not translational order; this is a fundamental characteristic feature of liquid crystals.

KEY WORDS: Hard rods; monomer-dimer model; nematic order; liquid crystal; convergence rate for renewal sequences.

1. INTRODUCTION AND RESULTS

In 1949, Lars Onsager proposed a theory of the isotropic-nematic phase transition in liquid crystals, which relied on the following simple heuristics [6]. Picture each molecule as a (very) long, (very) thin rod. There is no energetic interaction between the rods, except for hard-core exclusion. Since at low densities the molecules are typically far from each other, the resulting state will be an isotropic gas. However, at large densities it might be more favorable for the molecules to align spontaneously, since the resulting loss of orientational entropy is by far compensated by the gain of translational entropy: indeed, there are many more ways of placing nearly aligned rods than randomly oriented ones.

This is probably the first example of an entropy-driven phase transition. It shows that an increase of entropy can sometimes result in an

[1] Faculty of Industrial Engineering, Technion Haifa 32000, Israel; e-mail: ieioffe@ie.technion.ac.il
[2] UMR-CNRS 6085, Université de Rouen F-76821 Mont Saint Aignan, France; e-mail: Yvan.Velenik@univ-rouen.fr
[3] Faculty of Mathematics and Physics, Charles University 18600 Praha 8, Czech Republic; e-mail: mzahrad@karlin.mff.cuni.cz

apparently more ordered structure, hence the often used expression "order from disorder."

In spite of the obvious physical relevance of such issues, rigorous results are still very scarce. The only proof of such a phase transition has been given (as a side remark) in [1] for the following simple model: The rods are one-dimensional unit-length line segments in $\mathbb{R}^2$, with two possible orientations (say, horizontal and vertical); a configuration of N rods (N is not fixed) is specified by a family $(x, \sigma) \in \mathbb{R}^{2N} \times \{-1, 1\}^N$, where (x_{2k-1}, x_{2k}) is the position of the middle of the kth rod, while σ_k represents its orientation. A configuration in a subset V of $\mathbb{R}^2$ is admissible if all rods are inside V and are disjoint; let $\mathcal{C}_V$ denote this event. Then the measure describing the process is $\nu_\lambda(\,\cdot \mid \mathcal{C}_V)$, where ν_λ is the product of the Poisson point process in $\mathbb{R}^2$ of intensity $\lambda > 0$ and the Bernoulli process of parameter $\frac{1}{2}$. The main result is then that, in the thermodynamic limit $V \nearrow \mathbb{R}^2$, there are (at least) two limiting Gibbs states with long-range orientational order, for all λ large enough. This model has a very special feature though, namely two horizontal (respectively. vertical) rods have 0-probability of intersecting under ν_λ. This is a considerable simplification, and therefore this result does not provide any information on the much more interesting case of rods of finite width.

The other rigorous results are concerned with lattice versions of this problem. Heilmann and Lieb proved in a classical paper [3] that there is *no* phase transition in the monomer-dimer model, in the sense that the corresponding free energy is always analytic. This model is defined as follows (on $\mathbb{Z}^2$, their result holds more generally). Let V be a finite subset of $\mathbb{Z}^2$. For $x, y \in \mathbb{Z}^2$, we write $x \sim y$ if $|x - y| = 1$. Let $\mathcal{E}_V = \{\{x, y\} \subset v : x \sim y\}$; for $e, e' \in \mathcal{E}_V$ we write $e \sim e'$ if $e \cap e' \neq \emptyset$. The state space is given by $\Omega = \{0, 1\}^{\mathcal{E}}_V$. A configuration $\omega \in \Omega$ is admissible if $e \sim e' \Rightarrow \min(\omega_e, \omega_{e'}) = 0$. The probability of a configuration ω is then given by

$$\mu_\lambda(\omega) \propto \mathbf{1}_{\{\omega \text{ is admissible}\}}\, \lambda^{|\omega|},$$

where $|\omega| = \sum_{e \in \mathcal{E}_V} \omega_e$, and $\lambda > 0$. Informally, when a pair of sites e is such that $\omega_e = 1$, then the two sites are occupied by a dimer; a configuration is admissible if no site belongs to more than one dimer; λ is the dimers' fugacity.

An alternative approach to this model was then discovered by van den Berg [4]. Using disagreement percolation methods, he was able to give a very simple proof of the much stronger result that this model is in fact completely analytic, in the sense of [5]. This paper is also interesting in that it clearly points out the very special nature of dimers. Indeed, it would be impossible to push the analysis to arbitrary values of fugacities, were it not for a magical property of dimers: A monomer-dimer model on a graph G is actually equivalent to a pure hard-core gas on the line-graph of G.

Several other similar models have been introduced (see, e.g., [7, 8, 9]), in which existence of orientationally ordered states has been proved. All these models, however, share the same defect, namely the ordered states also automatically display long-range translational order, i.e. they are perturbations of periodic configurations. Thus they really can't be considered satisfactory models of liquid crystals, since a central characteristic of the latter is the liquid-like spatial behavior in the ordered states. In order to solve this problem, Heilmann and Lieb [2] proposed five different models of hard-core particles (actually dimers). For these models they proved the existence of long-range orientational order at low temperatures, and gave quite plausible arguments in favour of the absence of long-range translational order. These, however, were not pure hard-core models, since an additional attractive interaction favouring alignment of the dimers was introduced, and thus the question of whether pure hard-core interaction can give rise to such phases was left open (actually, Heilmann and Lieb even stated that it was "doubtful [...] whether hard rods on a cubic lattice without any additional interaction do indeed undergo a phase transition").

To the best of our knowledge, these are the only rigorous results pertaining to this problem. It would be extremely desirable to prove the existence of a phase transition in the monomer–k-mer model (replacing dimers above by k-mers, i.e. families of k aligned nearest-neighbor sites), for large enough k. This seems rather delicate however, and in this work we concentrate on another variant of the monomer-dimer model, with only hard-core exclusion and for which it is actually possible to prove existence of phases with orientational long-range order and no translational long-range order; actually it also seems possible to treat the three-dimensional case, which presumably would lead to a counterexample to the above claim. We hope to return to the monomer–k-mer problem and to the case of higher dimensions in the future.

Our model is defined as follows. We call rod a family of k, $k \in \mathbb{N}$, distinct, aligned, nearest-neighbor sites of $\mathbb{Z}^2$ a k-rod is a rod of length k, and we refer to 1-rod as vacancies. Let $V \subset \mathbb{Z}^2$; a configuration ω of our model inside V is a partition of V into a family of disjoint rods. We write $N_k(\omega)$ the number of k-rods in ω. The probability of the configuration ω is given by

$$\mu_{q,N,V}(\omega) \propto \mathbf{1}_{\{N_k(\omega)=0,\, \forall k>N\}}(2q)^{N_1(\omega)}\, q^{\sum_{k=2}^{N} N_k(\omega)}\,, \tag{1.1}$$

where $q > 0$ and $N \in \mathbb{N}$. Informally, only rods of length at most N are allowed; the activity of each rod of length at least 2 is q, and is independent of the rod's length; there is an additional activity $2q$ for vacancies.

Remark 1.1. The activity of vacancies can be removed at the cost of introducing an additional factor $(2q)^{-k}$ for each k-rods ($k \geq 2$).

Our main task is the proof of the following theorem, which states that for large enough N, there is a phase transition from a unique (necessarily isotropic) Gibbs state at large values of q to several Gibbs states with long-range orientational order at small values of q, but *no* translational order. This is thus the first model, where such a behavior can be proved.

Theorem 1.2.

1. *For any* $N \geq 2$*, there exists* $q_0 = q_0(N) > 0$ *such that, for all* $q \geq q_0$ *there is a unique, isotropic Gibbs state.*
2. *For any* $q > 0$ *sufficiently small there exist* $N_0 = N_0(q)$*, such that for all* $N \geq N_0$ *there are two different extremal Gibbs states with long-range orientational order. More precisely, there exists a Gibbs state* $\mu^h_{q,N}$ *such that*

$$\mu^h_{q,N}(\,0 \text{ belongs to a horizontal rod}\,) > 1/2\,. \tag{1.2}$$

In the sequel we shall refer to the infinite volume Gibbs state $\mu^h_{q,N}$ as to the horizontal state. By symmetry the $\pi/2$ rotation of the latter gives the vertical Gibbs state $\mu^v_{q,N}$, which would statistically favour vertically oriented rods.

A funny consequence of the techniques we develop in order to prove Theorem 1.2 is the following result on a sampling of infinite volume horizontal and vertical states by the shapes of the family of finite volume domains:

Theorem 1.3. *Let* $\underline{k} = (k_1, k_2)$ *be two natural numbers. For* $n = 1, 2, \ldots$*, consider lattice boxes*

$$V^{\underline{k}}_n = [-k_1 n, \ldots, k_1 n] \times [-k_2 n, \ldots, k_2 n],$$

and let $\mu_{q,N,V^{\underline{k}}_n}$ *be the finite volume Gibbs state specified in (1.1). Then if* q *and* N *satisfy conditions of 2) of Theorem 1.2,*

$$\lim_{n\to\infty} \mu_{q,N,V^{\underline{k}}_n} = \mu^h_{q,N} \quad \textit{if } k_1 > k_2$$

and

$$\lim_{n\to\infty} \mu_{q,N,V^{\underline{k}}_n} = \mu^v_{q,N} \quad \textit{if } k_1 < k_2$$

Theorem 1.2 is proved by showing that, for N large enough, the model defined above is a small perturbation (in a suitable sense) of the "exactly solvable" case $N = \infty$. For the latter, the theorem takes the following form. Let $q_{\mathrm{C}} = 1/(2 + 2\sqrt{2})$.

Theorem 1.4.

1. *Let* $N = \infty$*. For all* $q \geq q_{\mathrm{C}}$ *there is a unique, isotropic Gibbs state.*

2. *For all $q < q_C$, there are (at least) 2 different extremal Gibbs states with long-range orientational order. More precisely, there exists a Gibbs state μ_q such that*

$$\mu_q(\, 0 \text{ belongs to a horizontal rod}\,) > 1/2\,.$$

2. AN EXACTLY SOLVABLE CASE: PROOF OF THEOREM 1.4

In this section, we show that the model obtained by setting $N = \infty$ is actually exactly solvable, since it can be mapped on the 2D Ising model.

We suppose that our system is contained inside a square box V of linear size L; we suppose that we have periodic boundary conditions. We want to partition V into two disjoint subsets corresponding to the regions occupied by horizontal and vertical rods respectively. This can be done easily once we have said what we do with vacancies. The trick is to split vacancies into two species, horizontal and vertical. Doing so, starting from a configuration ω of our original model, we obtain a family of $2^{N_1(\omega)}$ different configurations $\widetilde{\omega}_i$, $i = 1, \ldots, N_1(\omega)$. The probability of each such configuration is then taken to be

$$\mu_{q,N,V}(\widetilde{\omega}) \propto \mathbf{1}_{\{N_k(\omega)=0,\,\forall k>N\}}\, 2^{-N_1(\widetilde{\omega})}\, (2q)^{N_1(\widetilde{\omega})}\, q^{\sum_{k=2}^{\infty} N_k(\widetilde{\omega})} = q^{\sum_{k=1}^{\infty} N_k(\widetilde{\omega})}\,.$$

We can now partition $V = V_\mathrm{h} \vee V_\mathrm{v}$ into two disjoint subsets. Once these subsets are fixed, the problem is reduced to the study of one-dimensional partition functions; indeed each maximal connected horizontal piece of V_h can be filled by horizontal rods independently of what choice is made for the rest of the configuration, and similarly for vertical pieces of V_v.

When $N = \infty$ the one-dimensional partition functions can be computed exactly:

$$Z_n^{1\mathrm{D}} = \sum_{i=0}^{n-1} \binom{n-1}{i} q^{i+1} = q\ (1+q)^{n-1} = \frac{q}{1+q}\ (1+q)^n\ ,$$

where $Z_n^{1\mathrm{D}}$ is the 1D partition function in a box of length $n \geq 1$. Now observe that the exponentially decreasing term $(1+q)^n$ is actually irrelevant, since its total contribution to the weight of a partition $V_\mathrm{h} \vee V_\mathrm{v}$ is $(1+q)^{|V|}$, and is therefore independent of the partition. We thus see that this model possess the remarkable property that all its 1D partition functions are actually equal to $e^{-4\beta} \stackrel{\triangle}{=} q/(1+q)$, and therefore independent of n. It is then very easy to compute the total weight of a partition:

$$\text{weight}\,(V_\mathrm{h},\, V_\mathrm{v}) \propto e^{-2\beta|\underline{\gamma}|}\,,$$

where $\underline{\gamma} = (\gamma_1, \ldots, \gamma_m)$ is the set of contours of the partition, i.e. the set of all bonds of the dual lattice intersecting a bond between two nearest-neighbor sites belonging one to $V_{\rm h}$ and the other to $V_{\rm v}$; $|\underline{\gamma}|$ is the total length of the contours.

One thus observes that the weight of partitions are the same as those of the corresponding configuration of the 2D Ising model at inverse temperature β, in the box V with periodic boundary conditions. Now notice that $\beta(q_{\rm C}) = 1/2 \log(1+\sqrt{2}) = \beta_{\rm C}$, the critical inverse temperature of the 2D Ising model. It immediately follows that for $q \geq q_{\rm C}$, the corresponding Ising model is in the high-temperature phase, and therefore possesses a unique Gibbs state. Statement 1 of Theorem 1.4 follows immediately from the symmetry of the latter Gibbs state.

To prove statement 2. requires only a simple additional argument. For a given collection of rods $\widetilde{\omega}$, let us denote by $Z_{\rm h}$ the number of sites of $V_{\rm h}$ containing vacancies, and $N_h = |V_{\rm h}| - Z_{\rm h}$; similarly introduce $N_{\rm v}$ and $Z_{\rm V}$. When $q < q_{\rm C}$, the Ising model is in the low-temperature region; consequently,

$$\mathbb{E}_{\mu_{q,V}}[||V_{\rm h}| - |V_{\rm v}||] = \mathbb{E}_{\text{Ising},\beta,V}\Big[\Big|\sum_{x\in V}\sigma_x\Big|\Big] > cL^2\,,$$

with $c > 0$. Since $\mu_{q,V}(\,0 \text{ belongs to a horizontal rod}\,) = L^{-2}\,\mathbb{E}_{\mu_{q,V}}[N_{\rm h}]$, the conclusion now follows easily from

$$||V_{\rm h}| - |V_{\rm v}|| \leq |N_{\rm h} - N_{\rm v}| + |Z_{\rm h} - Z_{\rm v}|\,,$$

and $\mathbb{E}_{\mu_{q,V}}[|Z_{\rm h} - Z_{\rm v}|] < CL$, by the Central Limit Theorem.

Remark 2.1. A lot of additional information (e.g., on the critical behavior) can be extracted from this mapping to the 2D Ising model. We refrain from doing that here, since this is quite straightforward...

Remark 2.2. As it was pointed to one of us by Lincoln Chayes, in $N = \infty$ case the techniques of reflection positivity enable to treat a more general situation when the rod weights are given by

$$\lambda^{N_1(\omega)} \prod_k q^{N_k(\omega)}.$$

In the above notation the case we consider here corresponds to a specific choice $\lambda = 2q$. However, the reflection positivity argument does not go through when there is a finite collection of admissible rod lengths, $N < \infty$.

3. ASYMPTOTICS OF ONE-DIMENSIONAL PARTITION FUNCTIONS

Our next step is to show that the model with finite (but large) N is actually a small perturbation of the exactly solvable model analyzed in Section 2. The idea, which is described in details in Subsection 4 is to replace all the 1D partition functions by their limiting values (for $n \to \infty$), and to expand the error term. To be able to control this expansion, we need a very good control on the speed of convergence of these 1D partition functions. This is the aim of the current subsection.

Our results on exponential rate of convergence of renewal sequences are certainly not new (see e.g. [12] and the references therein). However, we would like to stipulate that a very important input for our expansion techniques is not only the corresponding decay exponent but also smallness of the prefactor in front of it, see our key estimates (3.26) and (3.27) below.

3.1. The Setup

We shall consider here a general case of non-negative rod activities $\{f_k\}$ which we shall view as a perturbation of the geometric distribution,

$$f_k = qp^{k-1} + \epsilon_k; \quad k = 1, 2, \ldots, \tag{3.3}$$

where $p + q = 1$ and the activities $\{f_k\}$ are normalized to furnish a probability distribution, that is

$$\sum_k \epsilon_k = 0 \tag{3.4}$$

The important assumptions are those on the smallness of the perturbation sequence $\{\epsilon_k\}$ with respect to the background geometric distribution $\{qp^{k-1}\}$:

Assumption A1 There exist $\delta < \infty$ and $\rho \in (1, p^{-1})$ such that

$$|\epsilon_k| \leq \delta\rho^{-k}, \quad k = 1, 2, \ldots. \tag{3.5}$$

Assumption A2 There exists $\alpha > 0$ sufficiently small such that,

$$\delta < \alpha(\rho - 1)^2.$$

Assumption **A1** is an essential one. On the other hand, Assumption **A2** is more technical and it merely reflects an intended compromise between giving a relatively simple proof and yet generating a whole family of examples where the entropy driven phase transition takes place. Notice that since $\rho < 1/p$, assumption **A2** in fact implies a bound on δ in terms of q:

$$\delta < \frac{\alpha}{p^2} q^2.$$

Given $\{f_k\}$ as in (3.3) above we use it to set up the renewal relation:

$$g_1 = 1 \qquad \text{and} \qquad g_n = \sum_{k=1}^{n-1} f_k g_{n-k} \qquad n = 2, 3, \ldots. \tag{3.6}$$

Define the generating function of the $\{f_k\}$ sequence as

$$\mathbb{F}(\xi) = \sum_{k=1}^{\infty} f_k \xi^k = \frac{q\xi}{1 - p\xi} + \sum_k \epsilon_k \xi^k \triangleq \mathbb{Q}(\xi) + \mathbb{E}(\xi).$$

Above $\mathbb{Q}$ is the generating function of the geometric distribution $\{qp^{k-1}\}$ and, accordingly, $\mathbb{E}$ is the generating function of $\{\epsilon_k\}$.

In the sequel we use the notation

$$\mathbb{D}_r(x) = \{z \in \mathbb{C} \,:\, |z - x| < r\}$$

for an open complex ball of radius r centered at x. By **A1**, $\mathbb{F}$ is analytic on $\mathbb{D}_\rho(0)$.

The generating function $\mathbb{G}$ of the $\{g_n\}$ sequence is defined and analytic in $\{z \in \mathbb{C} \,:\, |z| < 1\}$. Of course, $\mathbb{G}(z) = (1 - \mathbb{F}(z))^{-1}$ and, by usual manipulations of the renewal theory,

$$\lim_{n\to\infty} g_n = \lim_{z\to 1}(1 - z)\,\mathbb{G}(z) = \frac{1}{\mathbb{F}'_N(1)} = \frac{1}{1/q + \sum_k k\epsilon_k} \triangleq g. \tag{3.7}$$

3.2. The Representation Formula

For every $\nu \in (0, 1)$,

$$\begin{aligned} g_n - g &= \frac{1}{2\pi i} \oint_{D_\nu(0)} \left\{ \frac{d\xi}{\xi^n(1 - \mathbb{F}(\xi))} - \frac{d\xi}{\xi^n(1 - \xi)\mathbb{F}'(1)} \right\} \\ &= \frac{1}{2\pi i} \oint_{D_\nu} \frac{d\xi}{\xi^n} \left\{ \frac{(\mathbb{F}(\xi) - 1) - \mathbb{F}'(1)(\xi - 1)}{(\mathbb{F}(\xi) - 1)(\xi - 1)\mathbb{F}'(1)} \right\} \end{aligned} \tag{3.8}$$

Now,

$$(\mathbb{F}(\xi)-1)-\mathbb{F}'(1)(\xi - 1) = \left\{(\mathbb{Q}(\xi) - 1) - \mathbb{Q}'(1)(\xi - 1)\right\} + \left\{\mathbb{E}(\xi) - \mathbb{E}'(1)(\xi - 1)\right\}$$

As a result,

$$\begin{aligned} 2\pi i\mathbb{F}'(1)\,(g_n - g) &= \oint_{D_\nu} \frac{d\xi}{\xi^n} \left\{ \frac{(\mathbb{Q}(\xi) - 1) - \mathbb{Q}'(1)(\xi - 1)}{(\mathbb{F}(\xi) - 1)(\xi - 1)} \right\} \\ &+ \oint_{D_\nu} \frac{d\xi}{\xi^n} \left\{ \frac{\mathbb{E}(\xi) - \mathbb{E}'(1)(\xi - 1)}{(\mathbb{F}(\xi) - 1)(\xi - 1)} \right\} \\ &\triangleq \oint_{D_\nu} \frac{d\xi}{\xi^n} \left\{ \frac{(\mathbb{Q}(\xi) - 1) - \mathbb{Q}'(1)(\xi - 1)}{(\mathbb{F}(\xi) - 1)(\xi - 1)} \right\} + \oint_{D_\nu} \frac{d\xi}{\xi^n} \mathbb{U}_1(\xi), \end{aligned} \tag{3.9}$$

where

$$\mathbb{U}_1(\xi) = \frac{\mathbb{E}(\xi)/(\xi - 1) - \mathbb{E}'(1)}{\mathbb{F}(\xi) - 1}.$$

On the other hand, since the geometric distribution $\{qp^{k-1}\}$ generates (via the renewal relation) a constant sequence $\{q\}$,

$$\oint_{D_\nu} \frac{\mathrm{d}\xi}{\xi^n} \left\{ \frac{(\mathbb{Q}(\xi) - 1) - \mathbb{Q}'(1)(\xi - 1)}{(\mathbb{Q}(\xi) - 1)(\xi - 1)} \right\} \equiv 0.$$

Subtracting the above expression for zero from the first term on the right hand side of 3.9 we obtain

$$-\oint_{D_\nu} \frac{\mathrm{d}\xi}{\xi^n} \left\{ \frac{\big(\mathbb{Q}(\xi) - 1 - \mathbb{Q}'(1)(\xi - 1)\big)\,\mathbb{E}(\xi)}{(\mathbb{F}(\xi) - 1)(\mathbb{Q}(\xi) - 1)(\xi - 1)} \right\} \stackrel{\Delta}{=} \oint_{D_\nu} \frac{\mathrm{d}\xi}{\xi^n} \mathbb{U}_2(\xi). \tag{3.10}$$

Thus, with $\mathbb{U}_1$ and $\mathbb{U}_2$ being defined as in (3.9) and (3.10) above, the representation formula (3.8) reads as

$$2\pi i \mathbb{F}'(1)\,(g_n - g) = \oint_{D_\nu} \frac{\mathrm{d}\xi}{\xi^n} \left\{ \mathbb{U}_1(\xi) + \mathbb{U}_2(\xi) \right\}.$$

Assume that it so happens that both $\mathbb{U}_1$ and $\mathbb{U}_2$ are analytic in an open neighbourhood of $\mathbb{D}_R(0)$ for some $R > 1$. Then (3.8) implies that

$$|g_n - g| \le \left(\max_{|\xi|=R} |\mathbb{U}_1(\xi)| + \max_{|\xi|=R} |\mathbb{U}_2(\xi)| \right) \frac{1}{2\pi \mathbb{F}'(1) R^n}. \tag{3.11}$$

We shall represent $\mathbb{U}_1$ and $\mathbb{U}_2$ as ratios of two analytic functions, In Subsection 3.3 we derive a lower bounds on the denominators, whereas in Subsection 3.4 we derive the corresponding upper bound for the numerators. Eventually we shall pick $R = (1 + \rho)/2$ and the target bound on $(\max_{|\xi|=R} |\mathbb{U}(z)| + \max_{|\xi|=R} |\mathbb{U}_2(\xi)|)$ is formulated in Subsection 3.5. It should become clear there that assumptions **A1** and **A2** enable an appropriate control on the smallness of the latter max. Finally the case of uniform rod weights is worked out in detail in Subsection 3.6

3.3. Lower Bounds on the Denominators

By **A1** the function

$$\frac{\mathbb{F}(\xi) - 1}{\xi - 1} = \frac{1}{1 - p\xi} + \frac{\mathbb{E}(\xi)}{\xi - 1} \stackrel{\Delta}{=} \frac{1}{1 - p\xi} + \mathbb{V}(\xi)$$

is analytic in $\mathbb{D}_\rho(0)$.

Let us pick $\eta \in (0, \rho - 1)$, later on we shall settle down with the choice $\eta = (\rho - 1)/2$, but in principle all the estimates below could be further optimized. Since $1 + \eta < \rho < 1/p$,

$$\inf_{\xi \in \mathbb{D}_{1+\eta}(0)} \left| \frac{1}{1 - p\xi} \right| \ge \frac{1}{1 + (1 + \eta)p} \ge \frac{1}{2}. \tag{3.12}$$

It, therefore, remains to derive an appropriate upper bound on $|\mathbb{E}(\xi)/(\xi-1)| = |\mathbb{V}(\xi)|$. There are two cases to be considered:

Case 1. $\xi \in \mathbb{D}_{1+\eta}(0) \setminus \mathbb{D}_\eta(1)$. Then, by **A1**,

$$|\mathbb{V}(\xi)| \le \frac{\delta}{\eta}\sum_1^\infty \left(\frac{1+\eta}{\rho}\right)^k = \frac{\delta(1+\eta)}{\eta(\rho-(1+\eta))}. \tag{3.13}$$

Case 2. $\xi \in \mathbb{D}_\eta(1)$. Since $\mathbb{E}(\cdot)$ is analytic in $\mathbb{D}_\eta(1)$,

$$\mathbb{E}(\xi) = \sum_1^\infty \epsilon_k \xi^k = \sum_1^\infty \epsilon_k \sum_{l=0}^k \binom{k}{l}(\xi-1)^l = \sum_1^\infty \tilde{\epsilon}_l(\xi-1)^l,$$

where we have used $\sum \epsilon_k = 0$ and, accordingly, have defined

$$\tilde{\epsilon}_l = \sum_{k=l}^\infty \epsilon_k \binom{k}{l}.$$

In view of the assumption **A1**,

$$|\tilde{\epsilon}_l| \le \delta \sum_{k=l}^\infty \rho^{-k}\binom{k}{l} = \delta\frac{\rho^{-l}}{(1-1/\rho)^{l+1}} = \frac{\delta\rho}{(\rho-1)}(\rho-1)^{-l}. \tag{3.14}$$

Consequently,

$$|\mathbb{V}(\xi)| \le \frac{\delta\rho}{(\rho-1)}\sum_1^\infty \frac{\eta^{l-1}}{(\rho-1)^l} = \frac{\delta\rho}{(\rho-1)(\rho-(1+\eta))}, \tag{3.15}$$

whenever $\xi \in \mathbb{D}_\eta(1)$.

Pick $\eta = (\rho-1)/2$. Then the right hand sides of both (3.13) and (3.15) are bounded above by $2\delta(1+\rho)/(\rho-1)^2$. Only at this stage we evoke assumption **A2**: under an appropriate choice of α the latter expression is as small as desired, say less than $1/6$. In view of (3.12,) (3.13) and (3.15) we, therefore, conclude:

Lemma 3.1. *Assume **A1** and **A2**, Then,*

$$\min_{\xi\in\mathbb{D}_{(1+\rho)/2}(0)} \left|\frac{\mathbb{F}(\xi)-1}{\xi-1}\right| \ge \frac{1}{3}. \tag{3.16}$$

Finally, by direct computation:

$$\left|\frac{\mathbb{Q}(\xi)-1}{\xi-1}\right| = \left|\frac{1}{1-p\xi}\right| \ge \frac{1}{1+(1+\eta)p} \ge \frac{1}{2}. \tag{3.17}$$

3.4. Upper Bound on the Numerators

We continue to employ the notation of the preceding subsection. In particular,

$$\mathbb{V}(\xi) \triangleq \frac{\mathbb{E}(\xi)}{1-\xi} \quad \text{and} \quad \frac{\mathbb{E}(\xi)}{1-\xi} - \mathbb{E}'(1) = \mathbb{V}(\xi) - \mathbb{V}(1).$$

Since by (3.4), $\sum \epsilon_k = \mathbb{E}(1) = 0$, $\mathbb{V}$ is analytic on $\mathbb{D}_\rho(0)$. As in Subsection 3.3 pick $\eta \in (0, \rho - 1)$ and consider the following two cases:

Case 1. $\xi \in \mathbb{D}_{1+\eta}(0) \setminus \mathbb{D}_\eta(1)$. By (3.13) and (3.15)

$$\left|\frac{\mathbb{V}(\xi) - \mathbb{V}(1)}{\xi - 1}\right| \le \frac{\delta\rho}{\eta(\rho-1)(\rho-(1+\eta))} + \frac{\delta(1+\eta)}{\eta^2(\rho-(1+\eta))} \le \frac{2\delta\rho}{\eta^2(\rho-(1+\eta))}.$$

Case 2. $\xi \in \mathbb{D}_\eta(1)$. Since $\mathbb{V}$ is analytic on $\mathbb{D}_\eta(1)$ we , employing the notation of Subsection 3.3, estimate:

$$\left|\frac{\mathbb{V}(\xi) - \mathbb{V}(1)}{\xi - 1}\right| = \left|\sum_{l=2}^{\infty} \tilde{\epsilon}_l (\xi - 1)^{l-2}\right| \le \frac{\delta\rho}{(\rho-1)^2(\rho-(1+\eta))},$$

where we have performed a straightforward series summation bounding $|\tilde{\epsilon}_l|$ as in (3.14).

Picking $\eta = (\rho - 1)/2$ and $R = (1 + \rho)/2$ we infer:

Lemma 3.2. *Assume* ***A1****, Then,*

$$\max_{\xi \in \mathbb{D}_R} \left|\frac{\mathbb{E}(\xi) - (\xi - 1)\mathbb{E}'(1)}{(\xi - 1)^2}\right| \le \frac{2\delta\rho}{(\rho-1)^3} \tag{3.18}$$

On the other hand,

$$\frac{1}{\xi - 1}\left(\frac{\mathbb{Q}(\xi) - 1}{\xi - 1} - \mathbb{Q}'(1)\right) = \frac{p}{q(1 - p\xi)}.$$

Since by assumption **A1**, $\max_{|\xi|=R} |\mathbb{E}(\xi)| \le \delta\rho/(\rho - 1)$, we arrive to the following bound for the numerator of $\mathbb{U}_2$:

$$\max_{|\xi|=R} \left|\frac{(\mathbb{Q}(\xi) - 1 - \mathbb{Q}'(1)(\xi - 1))\,\mathbb{E}(\xi)}{(\xi - 1)^2}\right| \le \frac{p}{q(1 - Rp)}\frac{\delta\rho}{(\rho - 1)} \le \frac{2\delta\rho}{p(\rho-1)^3}. \tag{3.19}$$

3.5. The Target Bound on $|r_n| = |g_n - g|$

As before set $R = (1 + \rho)/2$. By the estimates of Lemma 3.1 and Lemma 3.2,

$$\max_{|\xi \le R|} |\mathbb{U}_1(\xi)| + \max_{|\xi \le R|} |\mathbb{U}_2(\xi)| \le \frac{6\delta\rho(2 + p)}{p(\rho - 1)^3}.$$

Finally, $\mathbb{F}'(1) = 1/q + \sum k\epsilon_k$. By assumption **A1**,

$$\left|\sum_k k\epsilon_k\right| \leq \delta \sum_1^\infty kp^k = \frac{\delta p}{q^2} \leq \frac{\delta}{\rho - 1}\frac{1}{q}.$$

By the scaling relation between ρ and δ (assumption **A2**) the right hand side above is $o(q)$. In particular,

$$g = q(1 + o(1)), \tag{3.20}$$

as it now follows from (3.7).

Substituting the above estimates into (3.11):

Theorem 3.3. *Assume **A1** and **A2**. Set $R = (1 + \rho)/2$. Then,*

$$\begin{aligned} |r_n| = |g_n - g| \ &\leq \ \frac{12q\delta\rho(2+p)}{p(\rho-1)^3}R^{-n} = \frac{12q\delta\rho(2+p)}{p(\rho-1)^3}\left(\frac{2}{1+\rho}\right)^n \\ &\stackrel{\Delta}{=} c_1(q,\rho)\delta\left(\frac{2}{1+\rho}\right)^n. \end{aligned} \tag{3.21}$$

In particular for every $\nu < (\rho - 1)/4$,

$$\begin{aligned} \sum_n |r_n|(1+\nu)^n = \sum_n |g - g_n|(1+\nu)^n &\leq \frac{12\rho(2+p)(1+\rho)}{p} \\ \times \frac{q\delta}{(\rho - 1 - 2\nu)(\rho-1)^3} &\leq \frac{48\rho(2+p)(1+p)}{p(\rho-1)^4}\delta g \stackrel{\Delta}{=} c_2(q,\rho)\delta g. \end{aligned} \tag{3.22}$$

3.6. Uniform rod activities

Let the rod activities be given by

$$f_k = \begin{cases} \bar{q}; & k = 1, \ldots, N \\ 0; & \text{otherwise} \end{cases} \tag{3.23}$$

Above $\bar{q} = (\sum_1^N p^{k-1})^{-1} = (1-p)/(1-p^N)$. Thus, $\bar{q} - q = qp^N/(1-p^N)$. In other words the sequence of weights $\{f_k\}$ in (3.23) corresponds, in the notation of (3.3), to

$$\epsilon_k = \begin{cases} \frac{qp^N}{1-p^N}p^{k-1}; & k \leq N \\ -qp^{k-1}; & k > N \end{cases} \tag{3.24}$$

Without loss of generality we may assume that $q < 1/2$. Then for each N fixed the weights $\{\epsilon_k\}$ satisfy assumption **A1** with, for example,

$$\rho = \left(1 + \frac{1}{p}\right) \Big/ 2 = 1 + \frac{q}{2(1-q)} \quad \text{and} \quad \delta = \delta_N(q) = \left(1 - \frac{q}{2}\right)^N. \tag{3.25}$$

Of course, assumption **A2** will be also satisfied for such choice of ρ and δ as soon as

$$\delta_N(q) = \left(1 - \frac{q}{2}\right)^N \leq \alpha(\rho - 1)^2 = \alpha\left(\frac{q}{2(1-q)}\right)^2.$$

For the value of ρ related to q as in (3.25) set

$$\bar{c}_1(q) = c_1(q, \rho) \quad \text{and} \quad \bar{c}_2 = c_2(q, \rho),$$

where c_1 and c_2 are the universal constant which appear on the right hand sides of (3.21) and (3.22). Let us reformulate the claim of Theorem (3.3) as applied to the case of uniform rod activities (with the scaling choice (3.25) in mind):

Lemma 3.4. *Let $q < 1/2$ be fixed. Then there exists $N_0 = N_0(q)$ such that for every $N \geq N_0$, the uniform rod weights $\{f_k\}$ in (3.23) generate the renewal sequence $\{g_n\}$ which satisfies:*

$$|r_n| = |g_n - g| \leq \bar{c}_1(q)\delta_N(q)\left(1 + \frac{q}{4(1-q)}\right)^{-n}, \tag{3.26}$$

where g was defined in (3.7). Moreover, for $\nu = (\rho - 1)/4 = q/8(1-q)$,

$$\sum_n |r_n|\,(1+\nu)^n \;\leq\; \bar{c}_2(q)\delta_N(q)g. \tag{3.27}$$

In the sequel we shall actually need the following consequence of Lemma 3.4: Let $q < 1/2$ be fixed and let $\nu = q/8(1-q)$. Then for every $\epsilon > 0$ one can find $N_0 = N_0(q, \epsilon)$, such that

$$\sum_n n^2\sqrt{|r_n|}(1+\nu)^n \;\leq\; \epsilon, \tag{3.28}$$

for every $N \geq N_0$.

The role of the quantities $\sqrt{|r_n|}$ will become apparent in the over-counting argument which is employed in the proof of Lemma 4.2 below. There we shall split each excited interval of length n into two links on the dual lattice of weight $\sqrt{|r_n|}$ each.

4. PERTURBATION THEORY

In this section V is the lattice torus of a fixed (large) linear size L; $V = \mathbb{Z}^2/\mathrm{mod}(L)$. Notice, however, that all the estimates below do not depend on L.

4.1. Super-Contours

We proceed similarly as in the proof of Theorem 1.4. We first split vacancies into two families, and partition the box V into the two disjoint sub-boxes $V_{\rm h}$ and $V_{\rm v}$ containing the horizontal, resp. vertical, sites. Associated to this partition, there is a family of one-dimensional boxes $\underline{\Delta} = (\Delta_i)$, each of which is either a horizontal "segment" in $V_{\rm h}$, or a vertical "segment" in $V_{\rm v}$. The weight of the partition can then be expressed as a product over all $\Delta \in \underline{\Delta}$ of the corresponding one-dimensional partition functions $g_{|\Delta|} = Z^{\rm 1D}_{|\Delta|}$. Contrarily to what happens in the case considered in Theorem 1.4, these one-dimensional partition function *do* generally depend on the length of the corresponding box Δ. However, as we have seen in Section 3, these partition functions approach their limiting value g rather quickly, provided we choose N large enough. It is therefore convenient to expand them around this limiting value:

$$\prod_{\Delta\in\underline{\Delta}} g_{|\Delta|} = \prod_{\Delta\in\underline{\Delta}} g\left(1+\frac{g_{|\Delta|}-g}{g}\right). \tag{4.29}$$

We want to use this expansion in order to obtain a perturbation of the pure Ising model which appeared in the case of Theorem 1.4. Let us denote by $\underline{\gamma} = (\gamma_i)$ the family of Ising contours appearing when interpreting $V_{\rm h}$, resp. $V_{\rm v}$, as the region occupied by $+$, resp. $-$, spins. We can then associate to each of these contours a weight $w(\gamma) = e^{-2\beta|\gamma|}$, where we have set $e^{-2\beta} = \sqrt{g}$; this allows us to write simply

$$\prod_{\Delta\in\underline{\Delta}} g = \prod_{\gamma\in\underline{\gamma}} e^{-2\beta|\gamma|}.$$

We would like to encode all the information from the partition into these contours; in order to do this, we suppose that these contours come with a "color," i.e. each contour γ carries the information on which of the two sets $V_{\rm h}$ or $V_{\rm v}$ belong to which side of the contour. Of course, there is then a compatibility condition on these contours (in addition to their being disjoint): the colors must match.

We also introduce the set of excited intervals $\underline{I} = (I_i) \subset \underline{\Delta}$, and associate to such objects the weight $w(I) = (g_{|I|} - g)/g$. Using this we can rewrite the right hand side of (4.29) as,

$$\prod_{\Delta\in\underline{\Delta}}\left(1+\frac{g_{|\Delta|}-g}{g}\right) = \sum_{\underline{I}\subset\underline{\Delta}}\prod_{I\in\underline{I}} w(I).$$

Of course, since our colored contours $\underline{\gamma}$ contain all the information on the partition, the family $\underline{\Delta}$ is actually completely determined by the contours.

We now introduce our basic notion of super-contours (see Fig. 1), which are maximal connected components of (colored) contours and excited intervals

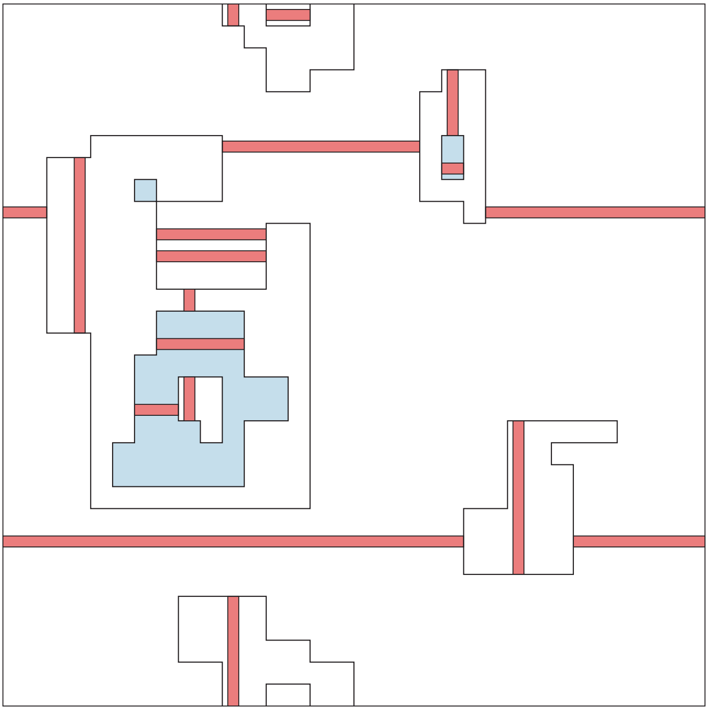

Fig. 1. A configuration of super-contours on the torus V. There are four super-contours. Notice that Γ is of horizontal type, but also possess some interior components of horizontal type (shaded in the picture). There are two super-contours winding around the torus; the contribution of configurations containing such super-contours being negligible when the box is large, as shown in Subsection 5.1, they can actually be neglected.

(saying that an interval is connected to a contour if at least one of the extremities of the interval belongs to the contour). We denote the family of super-contours by $\underline{\Gamma} = (\Gamma_i)$. The weight $w(\Gamma)$ of a super-contour Γ is then naturally given by the product of the weights of the contours and excited intervals it encompasses. We therefore finally obtain the following expression for the weight of the total partition function of our model:

$$Z_{q,N,L} = \sum_{\underline{\Gamma}} \prod_{\Gamma \in \underline{\Gamma}} w(\Gamma),$$

where the sum is taken over all compatible families of super-contours, i.e. those resulting from a partition $V = V_{\mathrm{h}} \vee V_{\mathrm{v}}$ in the way just described.

At this stage, it is not possible to apply a simple Peierls argument in order to control our model. Indeed, our super-contours are colored, and even though there is a symmetry in our model (under a simultaneous rotation by $\pi/2$ and exchange of horizontal and vertical sites. On the other hand, there is also a fundamental asymmetry: The shape of a region generally strongly favours one of the two species. This forces us to use the general strategy of the Pirogov-Sinai theory, which turns out to be quite simple in our case, due to the fact that, because of the above-mentioned symmetry, the free energies of the two phases are necessarily equal, and thus we are not required to add a suitable external field to reach phase coexistence.

The basic idea of the Pirogov-Sinai theory is to expand the partition function only over external contours, and introduce new weights, which reduce the compatibility condition to something of purely geometrical nature.

However, since we work on the lattice torus $V = \mathbb{Z}^2/\text{mod}(L)$, the notion of exterior of a contour is ambiguous. One way to mend the situation would be to fix a distinguished site, say 0, and to declare it to be "a point at infinity." On the other hand all our computations below are based on relatively crude combinatorial estimates which take into account local graph geometry of $\mathbb{Z}^2$, but not the global topological structure of V. Consequently, we shall from the start ignore (necessarily long) winding super-contours and then simply notice that should we use the "point at infinity" definition of exterior, the analog of (5.38) below would anyway render long winding contours improbable.

For any non-winding super-contour Γ the exterior of Γ is defined in a straightforward fashion and, accordingly, the type of a non-winding super-contour Γ will be declared to be horizontal or vertical if such is the colour of its exterior.

Thus, let $\mathcal{S}_L^h$ (respectively $\mathcal{S}_L^v$) be the set of all non-winding horizontal (respectively vertical) type super-contours on $V = \mathbb{Z}^2/\text{mod}(L)$. Of course, $\mathcal{S}_L^h$ and $\mathcal{S}_L^v$ are related by the $\pi/2$ rotation symmetry: If $\Gamma \in \mathcal{S}_L^h$, then $\theta_{\pi/2}\Gamma \in \mathcal{S}_L^v$. The interior $\text{int}(\Gamma)$ of coloured super-contours $\Gamma \in \mathcal{S}_L^h \cup \mathcal{S}_L^v$ is also coloured and in the sequel we shall write

$$\text{int}(\Gamma) = \text{int}_{\text{h}}(\Gamma) \cup \text{int}_{\text{v}}(\Gamma)$$

for the horizontal and vertical parts of $\text{int}(\Gamma)$. By the $\pi/2$-rotation symmetry of V, restricting to external contours of horizontal type ("h-type"), i.e. with their exterior colored as horizontal, yields exactly one-half of the full partition function: Using the notation $\underline{\mathcal{S}}_L^{h,\text{ext}}$ for set of all collections of compatible *external* contours from $\mathcal{S}_L^h$ and, accordingly, $\underline{\mathcal{S}}_L^h$ for for set of all collections of compatible contours from $\mathcal{S}_L^h$, we can then write

$$Z_{q,N,L} = 2 \sum_{\underline{\Gamma} \in \underline{\mathcal{S}}_L^{h,\text{ext}}} \prod_{\Gamma \in \underline{\Gamma}} w(\Gamma)\, Z^{\text{h}}_{\text{int}_{\text{h}}(\Gamma)} Z^{\text{v}}_{\text{int}_{\text{v}}(\Gamma)}$$

$$= 2 \sum_{\underline{\Gamma} \in \underline{\mathcal{S}}_L^{h,\mathrm{ext}}} \prod_{\Gamma \in \underline{\Gamma}} \widetilde{w}(\Gamma)\, Z^{\mathrm{h}}_{\mathrm{int}(\Gamma)}$$

$$= 2 \sum_{\underline{\Gamma} \in \underline{\mathcal{S}}_L^{h}} \prod_{\Gamma \in \underline{\Gamma}} \widetilde{w}(\Gamma)\,,$$

where the new weights are given by

$$\widetilde{w}(\Gamma) = w(\Gamma)\, \frac{Z^{\mathrm{v}}_{\mathrm{int_v}(\Gamma)}}{Z^{\mathrm{h}}_{\mathrm{int_v}(\Gamma)}}. \tag{4.30}$$

Notice that in the last expression the sum is over *all* compatible families of h-type super-contours; in particular, the compatibility condition is now purely geometrical.

4.2. Cluster Expansion

The next step is to show that the new weights are still under control. We first need a bit of terminology. Let us denote by $\Gamma \not\sim \Gamma'$ the relation "Γ is incompatible with Γ'". A cluster is a family $\mathcal{C}$ of super-contours which cannot be split into two disjoint families $\mathcal{C}_1$ and $\mathcal{C}_2$ such that all pairs $\Gamma \in \mathcal{C}_1$ and $\Gamma' \in \mathcal{C}_2$ are compatible. We also write $\mathcal{C} \not\sim \Gamma$ if there exists $\Gamma' \in \mathcal{C}$ such that $\Gamma' \not\sim \Gamma$. Finally we write $|\Gamma|$ for the total length of all the contours and intervals forming Γ. We want to be able to use the following classical sufficient condition for the convergence of the cluster expansion [11]:

Lemma 4.1. *Suppose that, for some small $a > 0$,*

$$\sum_{\Gamma' : \Gamma' \not\sim \Gamma} e^{2a\,|\Gamma'|} |\widetilde{w}(\Gamma')| \le a\,|\Gamma|\,, \tag{4.31}$$

for each Γ. Then $Z_{q,N,L} \neq 0$ and there exists a unique function Φ^{T} on the set of clusters such that

$$\log Z_{q,N,L} = \sum_{\mathcal{C} \subset V} \Phi^{\mathrm{T}}(\mathcal{C})\,.$$

Moreover,

$$\sum_{\mathcal{C} \not\sim \Gamma} |\Phi^{\mathrm{T}}(\mathcal{C})| e^{a\,|\mathcal{C}|} \le a\,|\Gamma|\,. \tag{4.32}$$

We claim that the weights $\widetilde{w}$ indeed satisfy (4.31) once q is chosen to be small enough and then N is chosen sufficiently large. The argument comprises two steps: First we shall check that (4.31) holds for the weights w. Next we shall argue that the conclusion of Lemma 4.1 for the weights w actually implies the validity of (4.31) for the target weights $\widetilde{w}$ for a possibly smaller value of q and larger values of N.

Lemma 4.2. *There exist $a > 0$, $q > 0$ and $N_0 = N_0(q)$ such that*

$$\sum_{\Gamma' : \Gamma' \not\sim \Gamma} e^{2a|\Gamma'|} |w(\Gamma')| \leq a\,|\Gamma|\,, \tag{4.33}$$

for every $N \geq N_0$.

Lemma 4.3. *There exist $a > 0$, $\widetilde{q} > 0$ and $\widetilde{N}_0 = \widetilde{N}_0(\widetilde{q})$ such that the weights $\widetilde{w}$ in 4.30 satisfy 4.31 for $\widetilde{q}$ and for every $N \geq \widetilde{N}_0$.*

4.3. Proof of Lemma 4.2

We shall actually prove a strengthened version of (4.33), which we proceed to describe. Set $w_0(\Gamma) = |w(\Gamma)|$. Any excited interval I of a super-contour Γ connects two dual bonds $b = (u, v)$ and $b' = (t, s)$ where both pairs $\{u, v\}$ and $\{t, s\}$ of dual vertices are recorded in the lexicographical order. Let us erase I and instead add two disjoint red links which connect between u and t. Notice that both red links lie on the dual lattice and have the same length $k = 1, 2, \ldots$ as the original excited intervals. We associate the weight $\sqrt{|r_k|}$ to each of these links.

Clearly after the above procedure is applied to all the excited intervals of Γ we end up with a connected edge self-avoiding polygon $\widehat{\Gamma} = b_1, \ldots, b_{2n}$ which entirely lies on the dual lattice and which consists of Ising nearest neighbour bonds with weights $\phi_b = \sqrt{g}$ and of red links with weights $\phi_b = \sqrt{|r_k|}$. With a slight abuse of notation we shall continue to write

$$w_0(\widehat{\Gamma}) = \prod_{b \in \widehat{\Gamma}} \phi_b$$

and $|\widehat{\Gamma}|$ for the weight and, respectively, for the total length of the Ising bonds and red links of $\widehat{\Gamma}$. Notice that $|\widehat{\Gamma}| > |\Gamma|$.

We claim that under the conditions of Lemma 4.2,

$$\sum_{\Gamma \ni 0} |\Gamma| e^{2a|\Gamma|} w_0(\Gamma) \leq a. \tag{4.34}$$

Pick (compare with (3.26))

$$2a = \frac{\rho - 1}{16} = \frac{q}{32(1 - q)}. \tag{4.35}$$

We over-count in the left hand side of (4.34) via ignoring geometric constraints: from each vertex of the dual lattice one is permitted to grow up bonds in all 4 possible directions: either usual Ising bonds of length 1 or red bonds of lengths $k = 1, 2, \ldots$. We should take into account all modified graphs $\widehat{\Gamma}$ which pass through the origin 0. It is convenient to assume that 0 is actually a vertex of $\widehat{\Gamma}$ and, accordingly, to multiply all the factors by k_1 – the length of the first link of $\widehat{\Gamma}$. Set $h_1 = \sqrt{g} + \sqrt{|r_1|}$ and $h_k = \sqrt{|r_k|}$ for $k \geq 2$. Notice that any modified graph $\widehat{\Gamma}$ contains at least four Ising bonds. Consequently, summing with respect to the number n of bonds and links of $\widehat{\Gamma}$ we estimate,

$$\sum_{\widehat{\Gamma} \ni 0} |\widehat{\Gamma}| e^{2a|\widehat{\Gamma}|} w_0(\widehat{\Gamma}) \leq \sum_{n=4}^{\infty} 4^n \sum_{k_1, \ldots, k_n} k_1 \left(\sum_1^n k_j \right) \mathrm{e}^{2a \sum_1^n k_j} \prod_1^n h_{k_j}. \tag{4.36}$$

Using a trivial inequality $k_1 (\sum_1^n k_j) \leq k_1^2 + 2^n \prod_1^n k_j$ we bound above the inner sum in (4.36) as

$$\sum_{k_1, \ldots, k_n} k_1 \left(\sum_1^n k_j \right) \mathrm{e}^{2a \sum_1^n k_j} \prod_1^n h_{k_j} \leq \left(\sum_1^{\infty} k^2 h_k \mathrm{e}^{2ak} \right) \left(\sum_1^{\infty} h_k \mathrm{e}^{2ak} \right)^{n-1} + 2^n \left(\sum_1^{\infty} k h_k \mathrm{e}^{2ak} \right)^n .$$

By (3.28) the latter expression is bounded above by $(\epsilon + \sqrt{g})^n (2^n + 1)$. In view of (3.20) and (4.35), $g^2 \ll a$. Therefore, (4.34) follows as soon as we choose ϵ (and hence $N_0(q, \epsilon)$ via the claim of Lemma 3.4) in such a way that

$$\sum_{n=4}^{\infty} 4^n (2^n + 1)(\epsilon + \sqrt{g})^n < a.$$

4.4. Proof of Lemma 4.3

Let q and N_0 be fixed as in the proof of Lemma 4.2, and let $\{w_0(\Gamma) \stackrel{\Delta}{=} |w(\Gamma)|\}$ be the absolute values of the weights of super-contours evaluated at such values of q and $N_0(q)$. It is enough to check that there is $\widetilde{q} \leq q$ and $\widetilde{N}_0 \geq N_0$, such that for every $N \geq \widetilde{N}_0$, the $(\widetilde{q}, N)$ super-contour weights $\{(w(\Gamma)\}$ satisfy:

$$|w(\Gamma)| e^{\sum_{\gamma \in \Gamma} |\Gamma|} \leq w_0(\Gamma) \quad \text{and} \quad \frac{Z_V^{\mathrm{v}}}{Z_V^{\mathrm{h}}} \leq e^{|\partial V|} \tag{4.37}$$

for every super-contour Γ and for each finite subset $V \subset \mathbb{Z}^2$.

Of course, only the second inequality in (4.37) deserves to be checked. This is done by induction on the volume. Obviously, if the volume $|V| = 1$, then we have $Z_V^{\mathrm{v}} / Z_V^{\mathrm{h}} = 1$. Suppose now that indeed

$$\frac{Z_V^{\mathrm{v}}}{Z_V^{\mathrm{h}}} \leq e^{|\partial V|},$$

for all $|V| < K$. We want to prove that this also holds when $|V| = K$. In order to see that, observe that all the super-contours appearing in these two partition functions have interiors of volume at most $K - 1$. Introducing the sets $\mathcal{S}^{\mathrm{h}}_{K-1}$ and $\mathcal{S}^{\mathrm{v}}_{K-1}$ of all clusters made up of h-type, resp. v-type, super-contours having (total) interior of volume at most $K - 1$, and using the symmetry present in the model, we can write

$$\frac{Z^{\mathrm{v}}_V}{Z^{\mathrm{h}}_V} = \frac{Z^{\mathrm{v}}_V}{\exp\left(\sum_{x\in V}\sum_{\mathcal{C}\in\mathcal{S}_{K^{\mathrm{v}}-1},\mathcal{C}\ni x}|\mathcal{C}\cap V|^{-1}\,\Phi^{\mathrm{T}}(\mathcal{C})\right)}\,\frac{\exp\left(\sum_{x\in V}\sum_{\mathcal{C}\in\mathcal{S}_{K-1},\mathcal{C}\ni x}|\mathcal{C}\cap V|^{-1}\,\Phi^{\mathrm{T}}(\mathcal{C})\right)}{Z^{\mathrm{h}}_V}.$$

Notice now that all the contours Γ appearing in the above partition functions have weights $\widetilde{w}(\Gamma)$ which, by the induction assumption and by the first of the inequalities in (4.37), satisfy:

$$|\widetilde{w}(\Gamma)| \le |w(\Gamma)|\,e^{\sum_{\gamma\in\Gamma}|\gamma|} \le w_0(\Gamma).$$

Therefore we can apply Lemma 4.1. Expanding the two partition functions and cancelling the terms involving clusters entirely contained inside V, we obtain the desired result, since by (4.32)

$$\frac{Z^{\mathrm{v}}_V}{Z^{\mathrm{h}}_V} \le e^{2a\,|\partial V|},$$

and $2a < 1$ once, according to (4.35), q is not very close to 1.

5. PROOFS OF THE MAIN RESULTS

In this section we complete the proofs of Theorem 1.2 and Theorem 1.3. As in the proof of Lemma 0 we proceed to work within the range of parameters (q, N) which satisfy (4.37).

5.1. Contribution of Long Super-Contours

As before let V be a lattice torus of linear size L. Given a supercontour $\Gamma \in \mathcal{S}^h_L$ define

$$\begin{aligned} G_{q,N,L}(\Gamma) &= \frac{1}{Z_{q,N,L}}\sum_{\underline{\Gamma}\ni\Gamma}\widetilde{w}(\underline{\Gamma}) \\ &= \frac{1}{2}\widetilde{w}(\Gamma)\,\exp\left(-\sum_{\mathcal{C}\not\sim\Gamma}\Phi^{\mathrm{T}}(\mathcal{C})\right). \end{aligned}$$

By (4.37) and (4.32),

$$\left|G_{q,N,L}(\Gamma)\right| \le w_0(\Gamma)\,e^{a|\Gamma|}.$$

Furthermore, by (4.31), there exist constants c_1 and c_2 such that

$$\sum_{|\Gamma|\geq k} w_0(\Gamma)e^{a|\Gamma|} \leq c_1 L^2 e^{-ak} \sum_{\Gamma\ni 0} w_0(\Gamma)e^{2a|\Gamma|}$$
$$\leq c_2 L^2 a e^{-ak}. \tag{5.38}$$

As a result, there exists $c_3 < \infty$, such that the contribution of super-contours $\Gamma \in \mathcal{S}_L^h$ with $|\Gamma| > c_3 \log L$ to the partition function $Z_{q,N,L}$ is, uniformly in L, negligible. The same argument applies, of course, in the case of vertical super-contours $\Gamma \in \mathcal{S}_L^v$.

5.2. Proof of Theorem 1.2

The first statement of Theorem 1.2 follows immediately from results of Gruber and Kunz [10].

Assume now that the parameters (q, N) satisfy (4.37). By the argument in the preceding subsection we may exclude long winding contours. Thus, the only thing remaining to be done in order to complete the proof of Theorem 1.2 is to estimate the probability that a given site, say 0, belongs to the interior of some short non-winding contour. In view of Lemma 4.1 the probability that 0 belongs to the interior of such a super-contour can then be written as

$$\sum_{\Gamma \circlearrowright 0} w(\Gamma) \frac{Z^{\mathrm{v}}_{\mathrm{int_v}(\Gamma)} Z^{\mathrm{h}}_{\mathrm{int_h}(\Gamma)}}{Z^{\mathrm{v}}_{q,N,V}} = \sum_{\Gamma \circlearrowright 0} \widetilde{w}(\Gamma) \exp\left(- \sum_{\mathcal{C} \not\sim \Gamma} \Phi^{\mathrm{T}}(\mathcal{C}) \right). \tag{5.39}$$

where $\Gamma \circlearrowright 0$ means that 0 is in the interior of the super-contour Γ. By (4.37) the latter expression is bounded above by

$$\sum_{\Gamma \circlearrowright 0} w_0(\Gamma)e^{a|\Gamma|} \leq \sum_{\Gamma \ni 0} |\Gamma| w_0(\Gamma) e^{a|\Gamma|},$$

The claim of the Theorem follows now from (4.34).

5.3. Infinite Volume States

Let $\mathcal{A}_\infty$ be the set of all such coverings $\widetilde{\omega}$ of $\mathbb{Z}^2$ by horizontal and vertical rods (we colour monomers as well), which contain only finite contours. Of course, for every $\widetilde{\omega}$ the notion of the exterior colour $\chi(\widetilde{\omega}) = h$ or v is well defined. By a straightforward application of Lemma 4.1:

Theorem 5.1. *There exists $q_0 > 0$ such that for every $q \leq q_0$ one can find $N_0 = N_0(q)$ which enjoys the following property: For every $N \geq N_0$ there exists an infinite volume Gibbs state $\mu^h_{q,N}$ (respectively $\mu^v_{q,N}$) such that*

$$\mu^h_{q,N}(\widetilde{\omega} \in \mathcal{A}_\infty; \chi(\widetilde{\omega}) = h) = 1 \tag{5.40}$$

(respectively $\mu^v_{q,N}(\mathcal{A}_\infty; \chi(\widetilde{\omega}) = v) = 1$*) . Furthermore, let* $\underline{\gamma}$ *be the (*random*) set of of all the exterior contours of* $\widetilde{\omega}$ *and, given a finite domain* $\Lambda \subset \mathbb{Z}^2$*, let* $\underline{\gamma}_\Lambda = (\gamma_1, \ldots, \gamma_n)$ *be a* fixed *compatible set of exterior contours, such that each* γ_k *intersects* Λ*,* $\Lambda \cap \gamma_k \neq \emptyset$*. Then,*

$$\mu^h_{q,N}(\underline{\gamma}_\Lambda \subset \underline{\gamma}) = \sum_{\underline{\Gamma}_\Lambda \sim \underline{\gamma}} \widetilde{w}(\underline{\Gamma}_\Lambda) \exp\left(- \sum_{\mathcal{C} \not\sim \underline{\Gamma}_\Lambda} \Phi^{\mathrm{T}}(\mathcal{C}) \right), \tag{5.41}$$

where the above sum is over all compatible collections $\underline{\Gamma}_\Lambda = (\Gamma_1, \ldots, \Gamma_m)$ *of super-contours satisfying:*

$$\forall\, l = 1, \ldots, m\ \exists\, k \text{ such that } \gamma_k \in \Gamma_l \quad \text{and} \quad \cup\, \gamma_k \subseteq \cup \Gamma_l.$$

Formulas (5.41) and (4.32) readily imply that $\mu^h_{q,N}$ has an exponential clustering property: Given two disjoint boxes Λ_1 and Λ_2 and two fixed compatible collections $\underline{\gamma}_{\Lambda_1}$ and $\underline{\gamma}_{\Lambda_2}$ with $\underline{\gamma}_{\Lambda_k} \subseteq \Lambda_k;\ k = 1, 2$, the following bound holds:

$$\left| \log \frac{\mu^h_{q,N}\left(\gamma_{\Lambda_1} \subset \underline{\gamma}\,; \gamma_{\Lambda_2} \subset \underline{\gamma}\right)}{\mu^h_{q,N}\left(\gamma_{\Lambda_1} \subset \underline{\gamma}\right) \mu^h_{q,N}\left(\gamma_{\Lambda_2} \subset \underline{\gamma}\right)} \right| \leq c_1 |\Lambda_1| |\Lambda_2| e^{-ac_2 \mathrm{d}(\Lambda_1, \Lambda_2)}, \tag{5.42}$$

where $\mathrm{d}(\Lambda_1, \Lambda_2)$ is a distance (say l_1) and c_1 and c_2 are two positive constants which depend only on q and N. In particular, (5.41) and (5.42) imply that $\mu^h_{q,N}$ is actually the unique Gibbs state satisfying (5.40).

5.4. Boundary Surface Tension

Consider vertical and horizontal intervals

$$J^v_k = (1/2, 1/2) + \{(0,0), (0,1), \ldots, (0, k-1)\} \quad \text{and}$$
$$J^h_k = (1/2, 1/2) + \{(0,0), \ldots (k-1, 0)\}.$$

By construction, both J^v_k and J^h_k are linear segments on the dual lattice $(1/2, 1/2) + \mathbb{Z}^2$. Given a rod $I = (u_1, \ldots, u_n) \subset \mathbb{Z}^2$ let us say that I intersects J^v_k; $I \cap J^v_k \neq \emptyset$ if

$$J^v_k \cap \operatorname{int}\left(\cup_{k=1}^n B_1(u_k)\right) \neq \emptyset,$$

where $B_1(u) = u + [-1/2, 1/2] \times [-1/2, 1/2] \subset \mathbb{R}^2$ and for a bounded set $A \subset \mathbb{R}^2$ the symbol $\operatorname{int}(A)$ stands for its $\mathbb{R}^2$-interior. In a similar fashion we define $I \cap J^h_k \neq \emptyset$. Notice that monomers cannot intersect J^v_k or J^h_k. Also, with such a definition, J^v_k cannot be intersected by a vertical rod and, accordingly, J^h_k cannot be intersected by a horizontal one.

Given a $\mathbb{Z}^2$ tiling $\widetilde{\omega} \in \mathcal{A}_\infty$ let us say that the event $\{J_k^v \cap \widetilde{\omega} = \emptyset\}$ (respectively $\{J_k^h \cap \widetilde{\omega} = \emptyset\}$) occurs if J_k^v (respectively J_k^h) does not intersect any of the rods of $\widetilde{\omega}$.

We define two types of boundary surface tensions:

$$\tau_{q,N} = -\lim_{k\to\infty} \frac{1}{k} \log \mu_{q,N}^h \left(J_k^v \cap \widetilde{\omega} = \emptyset\right) = -\lim_{k\to\infty} \frac{1}{k} \log \mu_{q,N}^v \left(J_k^h \cap \widetilde{\omega} = \emptyset\right), \tag{5.43}$$

and

$$\xi_{q,N} = -\lim_{k\to\infty} \frac{1}{k} \log \mu_{q,N}^h \left(J_k^h \cap \widetilde{\omega} = \emptyset\right) = -\lim_{k\to\infty} \frac{1}{k} \log \mu_{q,N}^v \left(J_k^v \cap \widetilde{\omega} = \emptyset\right), \tag{5.44}$$

In both cases the fact that the corresponding quantities are well defined follows from standard sub-additivity arguments based on the exponential clustering property (5.41) and on the $\pi/2$-rotational symmetry between the vertical and horizontal states.

Lemma 5.2. *For any q sufficiently small there exists $N_0 = N_0(q)$ such that for every $N \geq N_0$,*

$$\tau_{q,N} > \xi_{q,N}. \tag{5.45}$$

Proof: Since we do not try to prove the lemma in the whole range of entropy driven symmetry breaking, the poof boils down to a crude perturbative argument. We start with a lower bound on $\tau_{q,N}$: Fix $\gamma_1, \dots, \gamma_n$ to be the set of all exterior contours of $\widetilde{\omega}$ which intersect J_k^h. By (3.20) and (3.21),

$$\mu_{q,N}^v\left(\widetilde{\omega} \cap J_k^h = \emptyset \mid \gamma_1, \dots, \gamma_n\right) \leq (2q)^{|J_k^h \setminus \cup \gamma_l|}. \tag{5.46}$$

It remains, therefore, to derive an upper bound on

$$\mu_{q,N}^v\left(|J_k^h \setminus \cup_l \gamma_l| \leq k/2\right).$$

By a straightforward modification of the over-counting argument employed in the proof of Lemma 4.2 we infer from (5.41) that for a given collection $\gamma_1, \dots, \gamma_n$ of exterior contours,

$$\mu_{q,N}^v(\gamma_1, \dots, \gamma_n) \leq \exp\left(-2\beta \sum_1^n \operatorname{diam}(\gamma_l)\right),$$

where, as before, $e^{-2\beta} = \sqrt{g}$ and g is related to q via (3.20). Elementary combinatorics leads then to the following conclusion: If q is sufficiently small and

$N \geq N_0(q)$, then

$$\mu^v_{q,N}\left(|J^h_k \setminus \cup\gamma_l| \leq k/2\right) \;\leq\; \exp\left(-\frac{\beta k}{2}\right). \tag{5.47}$$

Combining (5.46) and (5.47) we arrive to the following lower bound on $\tau_{q,N}$:

$$\tau_{q,N} \;\geq\; -\frac{1}{8}\log q. \tag{5.48}$$

In order to derive a complementary upper bound on $\xi_{q,N}$ notice that on the level of events (under the vertical state $\mu^v_{q,N}$),

$$\left\{\widetilde{\omega} \cap J^v_k = \emptyset\right\} \;\supset\; \{\forall\,\gamma \text{ exterior contour of } \widetilde{\omega},\, J^v_k \cap \operatorname{int}(\gamma) = \emptyset\}.$$

Indeed, by the definition J^v_k can be intersected only by horizontal rods. Let us say that a super-contour Γ is intersection incompatible with J^v_k; $\Gamma \overset{i}{\not\sim} J^v_k$, if Γ contains a contour γ, such that $J^v_k \cap \operatorname{int}(\gamma) \neq \emptyset$. Then, by Lemma 4.1,

$$\mu_{q,N}\left(\widetilde{\omega} \cap J^v_k = \emptyset\right) \;\geq\; \exp\left(-\sum_{\mathcal{C}\overset{i}{\not\sim}J^v_k} |\Phi^{\mathrm{T}}(\mathcal{C})|\right) \;\geq\; e^{-ak}.$$

Consequently, $\xi_{q,N} \leq a$ and, in view of (5.48) and (4.35), the proof of Lemma 5.2 is concluded.

5.5. Sketch of a Proof of Theorem 1.3

Consider boxes $V^{\underline{k}}_n$ with periodic boundary conditions. As before we continue to ignore winding super-contours. In particular the notion of exterior colour is always well defined. Let, therefore, $Z^{h,\mathrm{per}}_{n,\underline{k}}$ and $Z^{v,\mathrm{per}}_{n,\underline{k}}$ be the partition functions of rod tilings of $V^{\underline{k}}_n$ with the exterior colour being fixed as h (respectively v). By (5.38) and Lemma 4.1,

$$\left|\log \frac{Z^{h,\mathrm{per}}_{n,\underline{k}}}{Z^{v,\mathrm{per}}_{n,\underline{k}}}\right| \;\leq\; c_3 n^2 e^{-c_4 a n}. \tag{5.49}$$

Finally, let $\mu^{h,\mathrm{per}}_{n,\underline{k}}$ and $\mu^{v,\mathrm{per}}_{n,\underline{k}}$ be the corresponding Gibbs states.

The partition functions $Z^{h,\mathrm{f}}_{n,\underline{k}}$ and $Z^{v,\mathrm{f}}_{n,\underline{k}}$ of the (exterior colour) horizontal and vertical tilings of $V^{\underline{k}}_n$ with *free* boundary conditions are related to $Z^{h,\mathrm{per}}_{n,\underline{k}}$ and $Z^{v,\mathrm{per}}_{n,\underline{k}}$ as follows: Set

$$J^v_{n,\underline{k}} = \left\{i = (i_1, i_2) \in V^{\underline{k}}_n \;:\; i_1 = 0\right\} \quad \text{and} \quad J^h_{n,\underline{k}} = \left\{i = (i_1, i_2) \in V^{\underline{k}}_n \;:\; i_2 = 0\right\}.$$

Then,

$$\frac{Z^{h,\mathrm{f}}_{n,\underline{k}}}{Z^{h,\mathrm{per}}_{n,\underline{k}}} = \mu^{h,\mathrm{per}}_{n,\underline{k}}\big(\widetilde{\omega} \cap J^{h}_{n,\underline{k}} = \emptyset\,;\, \widetilde{\omega} \cap J^{v}_{n,\underline{k}} = \emptyset\big),$$

and, respectively,

$$\frac{Z^{v,\mathrm{f}}_{n,\underline{k}}}{Z^{v,\mathrm{per}}_{n,\underline{k}}} = \mu^{v,\mathrm{per}}_{n,\underline{k}}\big(\widetilde{\omega} \cap J^{h}_{n,\underline{k}} = \emptyset\,;\, \widetilde{\omega} \cap J^{v}_{n,\underline{k}} = \emptyset\big).$$

By (5.43) and (5.44) the latter probabilities are logarithmically asymptotic to

$$\exp(-n((2k_2+1)\tau_{q,N} + (2k_1+1)\xi_{q,N})) \quad \text{and}$$
$$\exp(-n((2k_1+1)\tau_{q,N} + (2k_2+1)\xi_{q,N}))$$

respectively. The claim of Theorem 1.3 follows now from (5.49) and Lemma 5.2.

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REFERENCES

1. J. Bricmont, K. Kuroda and J. L. Lebowitz, The structure of Gibbs states and phase coexistence for nonsymmetric continuum Widom-Rowlinson models, *Z. Wahrsch. Verw. Gebiete* **67**(2), (1984) pp. 121–138.
2. O. J. Heilmann and E. H. Lieb, Lattice models for liquid crystals, *J. Statist. Phys.* **20**(6), (1979) pp. 679–693.
3. O. J. Heilmann and E. H. Lieb, Theory of monomer-dimer systems, *Comm. Math. Phys.* **25**, (1972) pp. 190–232.
4. J. van den Berg, On the absence of phase transition in the monomer-dimer model, Perplexing problems in probability, *Progr. Probab.* **44**, (Birkhäuser Boston, Boston, MA 1999) pp. 185–195.
5. R. L. Dobrushin and V. Warstat, Completely analytic interactions with infinite values, *Probab. Theory Related Fields* **84**(3), (1990) pp. 335–359.
6. L. Onsager, The effects of shape on the interaction of colloidal particles, *Ann. N. Y. Acad. Sci.* **51**, (1949) pp. 627–659.
7. D. A. Huckaby, Phase transitions in lattice gases of hard-core molecules having two orientations, *J. Statist. Phys.* **17**(5), (1977) pp. 371–375.
8. J. L. Lebowitz and G. Gallavotti, Phase transitions in binary lattice gases, *J. Math. Phys.* **12**, (1971) pp. 1129–1133.
9. O. J. Heilmann, Existence of phase transition in certain lattice gases with repulsive potential, *Lett. Nuovo Cim.* **3**, (1972) pp. 95–1xx.
10. C. Gruber and H. Kunz, General properties of polymer systems, *Comm. Math. Phys.* **22**, (1971) pp. 133–161.

11. R. Kotecký and D. Preiss, Cluster expansion for abstract polymer models, *Comm. Math. Phys.* **103**(3), (1986) pp. 491–498.
12. Kenneth S. Berenhaut and Robert Lund, Geometric renewal convergence rates from hazard rates, *J. Appl. Probab.* **38**(1), (2001) pp. 180–194.